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Loss of gravitational mass of a rotating rod as outgoing gravitational radiation

M A ROTENBERG

Division of Science, University of Wisconsin-Parkside, Kenosha, Wisconsin, USA

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Abstract. The solution of the gravitational field equations of general relativity is obtained for the metric field of a rotating rod. A Schwarzschildian correction term is found that represents a steady diminution of mass of the rod on account of the gravitational waves emitted.

1. Introduction

In studying gravitational waves from an isolated source of any form without special symmetry, Sachs (1962) found in his selected metric† an expression representing a monotonic decrease in mass of the source. The rate of this mass loss for a cohesive source was shown by Rotenberg (1971) to be equivalent to the rate at which energy is carried away by the waves, so that Sachs' result gives physical significance to these waves. To arrive at his result, Sachs used a method of approximation which had been previously introduced by Bondi for an axially and reflection symmetric source (Bondi 1962, Bondi *et al* 1962) and which involves the expansion of the metric tensor in negative powers of a suitably defined radial distance r. He then solved the empty-space field equations‡

$$R_{ik} = 0 \tag{1.1}$$

of general relativity by equating to zero coefficients of successive powers of r^{-1} in R_{ik}. Unfortunately, as pointed out by Bonnor and Rotenberg (1966), this method has one serious drawback: the expansion can be shown to contain terms diverging with time. In view of this, a different approximation method, one involving a double-series expansion of the type introduced by Bonnor (1959), is applied to the special case of a rotating rod to confirm that the rod steadily loses mass (in the second approximation to the field equations (1.1)) at the rate at which radiation energy is transferred from the rod. Comparing this result with the one mentioned above from Rotenberg (1971), we see that Sachs' expression for the mass decrease is substantiated, at least for the particular source.

In § 2 the spinning rod is described at considerable length. The double-series approximation method is presented in § 3, and the Sachs metric, selected for use by

† Henceforth referred to as the Sachs metric, of which the earlier axisymmetric version is the Bondi metric, invented by Bondi (1960).

‡ In this paper, a Latin index runs from 1 to 4, a Greek index from 1 to 3; the summation convention applies to both indices.

this method and resulting in drastic reduction in calculation, is exhibited in § 4. Like the Bondi metric, this one prevents the appearance of terms in $r^{-1} \ln r$ in its solution at any stage of approximation. Such terms would occur if harmonic coordinates were employed, as in a work by Fock (1959, § 87), and also in works by Clark (1941, 1947) related to the topic of this paper. The Sachs metric uses a different coordinate system and has the merit of satisfying the usual boundary conditions at spatial infinity. In § 5, the external solution of the linear approximation (to the field equations (1.1)) is derived, for the rotating rod, in coordinates of the Sachs metric. The corresponding solution of the second approximation, calculated in § 6, is shown to contain a Schwarzschildian correction term referring to a rate of loss in gravitational mass of the rod equal to the rate at which energy of radiation is emitted. The more complicated calculations are relegated to the Appendix.

2. The rotating rod

It would be convenient to provide the rod with some mechanism that could set it in spinning motion for a finite period of time after which the mechanism would arrest the motion. Then the Schwarzschildian term of order r^{-1} in the final stationary metric for the rod could be compared with that in the initial stationary metric to determine any secular diminution in mass suffered by the rod over the period of rotation.

For simplicity, let the mechanism be a uniform circular ring passing through two holes in the rod so situated that the centre of the ring coincides with the centre of mass of the rod, and let the common centre of mass be chosen as the origin O of a (pseudo) rectangular cartesian coordinate system Oxyz. Initially, the ring alone rotates in the xy plane about O with constant angular velocity ω_0, friction between the ring and the rod being so negligible that the rod practically remains in a stationary position. Then, shortly before $t = 0$ (at time $-t_0$, say), friction is set up between the ring and the rod by a clamping device, smoothly starting the rod in motion so that the entire system rotates about O at constant angular velocity $\omega < \omega_0$ from time $t = 0$, when the rod assumes the position chosen as the x axis, to time $t = T$. Finally, the rod is brought smoothly to rest (with the ring rotating at its former angular velocity ω_0 about O) during a short interval t_1: this is achieved by a separating recoil device.

It can be shown that outside the interval $-t_0 \leqslant t \leqslant T+t_1$ the system produces a stationary field, since the only moving component then is the ring, which rotates uniformly in its own constant position. Thus we have a rotating system emitting gravitational sandwich waves during the interval $-t_0 \leqslant t \leqslant T+t_1$, outside of which the field is stationary and can be proved to take a Schwarzschild form (such as the form (4.3) in the Sachs metric) up to order r^{-1}.

Although the proof of the foregoing is a subject for a future paper, it is well to justify here to a certain extent the assumption that a uniform rotating ring produces a field which is asymptotically Schwarzschildian. This is done by the following consideration: giving plausible reasons, Newman and Janis (1965) suggested that the ring may be represented by the Kerr metric, which we write in the form (see Boyer and Price 1965)

$$\begin{aligned} \mathrm{d}s^2 = {} & -\mathrm{d}r^2 - 2b \sin^2\theta \, \mathrm{d}r \, \mathrm{d}\phi - (r^2+b^2)\sin^2\theta \, \mathrm{d}\phi^2 - (r^2+b^2\cos^2\theta)\, \mathrm{d}\theta^2 \\ & + \mathrm{d}t^2 - \frac{2mr}{r^2+b^2\cos^2\theta}(\mathrm{d}r + b\sin^2\theta\,\mathrm{d}\phi + \mathrm{d}t)^2 \end{aligned} \tag{2.1}$$

where m is the mass and b the radius of the ring and where (r, θ, ϕ) are (pseudo) spherical polar coordinates of the field point P. Observing that

$$\frac{2mr}{r^2+b^2\cos^2\theta} = 2mr^{-1}+\mathrm{O}(r^{-3}) \tag{2.2}$$

and that contributions from the coefficients $g_{12}, g_{13}, g_{24}, g_{34}$ of the metric are divided by r and the coefficients g_{22}, g_{23}, g_{33} are divided by r^2 on transformation from coordinates (r, θ, ϕ) to coordinates (x, y, z) we find that, on neglect of expressions of order r^{-2} for large r, the metric (2.1) becomes

$$\begin{aligned} \mathrm{d}s^2 = {} & -\mathrm{d}r^2-2b\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\phi-r^2\sin^2\theta\,\mathrm{d}\phi^2-r^2\,\mathrm{d}\theta^2 \\ & +\mathrm{d}t^2-2mr^{-1}(\mathrm{d}r+\mathrm{d}t)^2. \end{aligned} \tag{2.3}$$

Carrying out the coordinate transformation of the form

$$\phi = \phi^*+bf(r) \qquad \text{with } f(r) \simeq r^{-1} \text{ for large } r \tag{2.4}$$

dropping the asterisks and ignoring terms of order r^{-2} for large r, we arrive at the following form of the Schwarzschild metric for a central particle of mass m:

$$\mathrm{d}s^2 = -\mathrm{d}r^2-r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)+\mathrm{d}t^2-2mr^{-1}(\mathrm{d}r+\mathrm{d}t)^2 \tag{2.5}$$

(see Rotenberg 1964 p 62, equation (62*a*), Boyer and Price 1965, equation following equation (1)). Thus the field of the rotating ring is asymptotically Schwarzschildian.

To conclude, we shall assume that any change in mass of the system due to the smooth transition of short duration from rest to motion for the rod or from motion to rest is negligible in comparison with the total loss in mass of the system during the spin of the rod, especially if it takes place for considerable time†. This assumption is justified towards the end of § 6, where it is also shown that a rod that continually rotates, without a mechanism to start and terminate its spin, likewise decreases its mass steadily on account of the waves emitted.

3. The double-parameter approximation method

We briefly describe below the method of approximation to be used in this paper. Invented by Bonnor (1959), this method may be applied to the external field of any finite coherent gravitating source.

Let m be the total mass of the source, so that, if T_{ik} is the material energy tensor, then in the linear approximation (to equation (1.1))

$$m = \int_V T_{44}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{3.1}$$

where V is any space volume enclosing the source and T_{44} is measured in pseudogalilean coordinates (x, y, z, t). Let a be any chosen constant having the dimension of length. Then the coefficients of the metric are to be expanded as doubly-infinite convergent series

$$g_{ik} = \overset{(00)}{g}_{ik}+\sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{g}_{ik} \tag{3.2}$$

† Accordingly, we shall refer to the period of rotation of the rod as $0 \leqslant t \leqslant T$, not precisely as $-t_0 \leqslant t \leqslant T+t_1$.

in ascending powers of the constant parameters m and a; $\overset{(00)}{g}_{ik}$ and $\overset{(ps)}{g}_{ik}$ are independent of m and a, and $\overset{(00)}{g}_{ik}$ constitute the flat space–time metric. (Cf Bonnor 1959, Rotenberg 1964 § 2.3, Bonnor and Rotenberg 1966.) It can be shown that the external solution of the linear approximation (to equation (1.1)) is the part of the expansion (3.2) *linear* in m, namely

$$g_{ik} = \overset{(00)}{g}_{ik} + \sum_{s=0}^{\infty} ma^s \overset{(1s)}{g}_{ik} \qquad \text{with } \overset{(11)}{g}_{ik} = 0 \tag{3.3}$$

(Rotenberg 1964 Appendix (A.1), 1968a, 1968b and 1971).

Substituting the expansion (3.2) into the field equations (1.1), picking out the coefficient of $m^p a^s$ from the result and equating it to zero, we obtain a set of ten second-order differential equations, called the (ps) *approximation*, which assume the forms

$$\Phi_{lm}(\overset{(ps)}{g}_{ik}) = \overset{(ps)}{\Psi}_{lm}(\overset{(qr)}{g}_{ik}) \qquad (q \leqslant p-1, r \leqslant s). \tag{3.4}$$

The left hand sides of these equations are linear in $\overset{(ps)}{g}_{ik}$ (and their derivatives), while the right hand sides are nonlinear in $\overset{(qr)}{g}_{ik}$ (and their derivatives), known from earlier approximation steps. Thus, any $(1s)$ approximation is linear and homogeneous in $\overset{(1s)}{g}_{ik}$ and their derivatives ($\overset{(1s)}{\Psi}_{lm} = 0$) and, consequently, pertains to the linear approximation. The complete set of $(1s)$ approximations ($s = 0, 1, 2, \ldots$), along with the trivial (00) approximation corresponding to flat space–time, make up the linear approximation. For $p \geqslant 2$, the (ps) approximation is nonlinear: the complete set of $(2s)$ approximations ($s = 0, 1, 2, \ldots$) constitutes the second approximation, the complete set of $(3s)$ approximations ($s = 0, 1, 2, \ldots$) forms the third approximation, and so on.

The solution of the (ps) approximation, which is represented by the $\overset{(ps)}{g}_{ik}$ satisfying equations (3.4), is also simply referred to as the (ps) *solution*. Alternative expressions denoting the $(1s)$ solution for $s \geqslant 1$ are 2^s *pole wave* (*solution*) and $(1s)$ *wave*; and the 2^s pole wave solutions, together with the static *monopole solution* represented by $\overset{(00)}{g}_{ik} + m\overset{(10)}{g}_{ik}$, form what is known as the *multipole wave solution* (of the linear approximation), (3.3). Evidently, from the second of equations (3.3), the dipole wave $\overset{(11)}{g}_{ik}$ is absent from the multipole wave solution. The leading wave-like terms in this solution are the *quadrupole* wave (involving ma^2), the *octupole* wave (involving ma^3) etc (Boardman and Bergmann 1959, Bonnor 1963, Rotenberg 1964, Appendix (A.1)).

It is a well known result that the 4-momentum of an isolated coherent gravitating source remains constant in the linear approximation (Rotenberg 1964, Appendix (A.2)). A steady change of 4-momentum may take place in the second approximation, and it is our object to show that there does occur a steady diminution in the mass of the rotating rod; this is achieved in § 6.

4. The Sachs metric

In solving the important approximations for the spinning rod we shall employ the metric of Sachs (1962); this may be presented in the form

$$\begin{aligned} ds^2 &= -r^2(B\, d\theta^2 - 2I \sin\theta\, d\theta\, d\phi + C \sin^2\theta\, d\phi^2) + D\, du^2 \\ &\quad + 2F\, dr\, du + 2Gr\, d\theta\, du + 2Jr \sin\theta\, d\phi\, du \\ BC - I^2 &= 1. \end{aligned} \tag{4.1}$$

In this, B, C, D, F, G, I, J are functions of the coordinates (r, θ, ϕ, u), which are defined as follows: r is a null coordinate representing something like the distance along a radial ray from the origin O; θ and ϕ are respectively the pseudopolar and pseudoazimuthal angles constant along a radial ray; and the time-like coordinate u, also constant along a radial ray, closely corresponds to the retarded time $t-r$ of flat space–time; thus the null coordinate curves r = variable (namely radial rays) generate the null coordinate hypersurfaces u = constant. The Sachs metric (4.1) is an extension of the axisymmetric metric of Bondi (1960, 1962) and represents fields with no special symmetry.

In coordinates of the Sachs metric, flat space–time is represented by

$$\mathrm{d}s^2 = -r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2) + \mathrm{d}u^2 + 2\, \mathrm{d}r\, \mathrm{d}u \tag{4.2}$$

and the external Schwarzschild space–time by

$$\mathrm{d}s^2 = -r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2) + (1 - 2mr^{-1})\, \mathrm{d}u^2 + 2\, \mathrm{d}r\, \mathrm{d}u \tag{4.3}$$

m being the total mass of the spherically symmetric source.

In accordance with the expansion (3.2), the coefficients of the metric (4.1) will have similar expansions given by

$$\begin{aligned}
-r^{-2}g_{22} &= B = 1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{B} \\
-r^{-2}\operatorname{cosec}^2\theta\, g_{33} &= C = 1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{C} \\
g_{44} &= D = 1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{D} \\
g_{14} &= F = 1 + \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{F} \\
r^{-1}g_{24} &= G = \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{G} \\
r^{-2}\operatorname{cosec}\theta\, g_{23} &= I = \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{I} \\
r^{-1}\operatorname{cosec}\theta\, g_{34} &= J = \sum_{p=1}^{\infty}\sum_{s=0}^{\infty} m^p a^s \overset{(ps)}{J}
\end{aligned} \tag{4.4}$$

in which $\overset{(ps)}{B}, \overset{(ps)}{C}, \ldots, \overset{(ps)}{J}$ are functions of (r, θ, ϕ, u), and $\overset{(ps)}{B}, \overset{(ps)}{C}$ and $\overset{(ps)}{I}$ are related by the second of equations (4.1). The notation (4.4) will henceforth be used.

5. The (1*s*) approximations

For the rotating rod we derive here suitable external solutions of the leading (1*s*) approximations, given, in coordinates (r, θ, ϕ, u) of the Sachs metric employed here, by equations (A.1) to (A.10) of the Appendix with $P = Q = \ldots = W = 0$. To save space we use the external multipole wave solution, of the linear approximation, obtained in these coordinates by Rotenberg (1971) for any bounded cohesive source with its centre of mass chosen as the origin O. Explicitly to the quadrupole wave contribution it is†

$$B = 1+\sum_{s=2}^{\infty} ma^s \overset{(1s)}{B} \qquad C = 1-\sum_{s=2}^{\infty} ma^s \overset{(1s)}{B}$$
$$D = 1-2mr^{-1}+\sum_{s=2}^{\infty} ma^s \overset{(1s)}{D} \qquad F = 1 \qquad G = \sum_{s=2}^{\infty} ma^s \overset{(1s)}{G} \tag{5.1}$$
$$I = \sum_{s=2}^{\infty} ma^s \overset{(1s)}{I} \qquad J = \sum_{s=2}^{\infty} ma^s \overset{(1s)}{J}$$

in which the monopole (10) solution is the sole surviving component

$$\overset{(10)}{D} = -2r^{-1} \tag{5.2}$$

the dipole (11) wave solution is nonexistent

$$\overset{(11)}{g_{ik}} = 0 \tag{5.3}$$

and the quadrupole (12) wave solution is given by

$$\begin{aligned}
\overset{(12)}{B} &= (n_{\sigma,2}n_{\rho,2}-\operatorname{cosec}^2\theta\, n_{\sigma,3}n_{\rho,3})(r^{-1}k''_{\sigma\rho}+r^{-3}k_{\sigma\rho})\\
\overset{(12)}{D} &= (-3n_\sigma n_\rho+\delta_{\sigma\rho})(2r^{-1}k''_{\sigma\rho}+2r^{-2}k'_{\sigma\rho}+r^{-3}k_{\sigma\rho})\\
\overset{(12)}{G} &= n_\sigma n_{\rho,2}\{2r^{-1}k''_{\sigma\rho}+2r^{-2}(-2k'_{\sigma\rho}+a_{\sigma\rho})-3r^{-3}k_{\sigma\rho}\}\\
\overset{(12)}{I} &= -2\operatorname{cosec}\theta\, n_{\sigma,2}n_{\rho,3}(r^{-1}k''_{\sigma\rho}+r^{-3}k_{\sigma\rho})\\
\overset{(12)}{J} &= \operatorname{cosec}\theta\, n_\sigma n_{\rho,3}\{2r^{-1}k''_{\sigma\rho}+2r^{-2}(-2k'_{\sigma\rho}+a_{\sigma\rho})-3r^{-3}k_{\sigma\rho}\}.
\end{aligned} \tag{5.4}$$

In equations (5.4) the notations are

$$n_\alpha \overset{\text{def}}{=} \frac{x_\alpha}{r} = \frac{(x, y, z)}{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{5.5}$$

a comma subscript $_{,2}$ or $_{,3}$ denotes differentiation with respect to θ or ϕ, respectively; the quantities

$$k_{\alpha\beta} \overset{\text{def}}{=} m^{-1}a^{-2}\int_V x_\alpha x_\beta T_{44}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{5.6}$$

(V being any space volume containing the source and T_{44} being measured in pseudogalilean coordinates $x_i = (x, y, z, t)$) are to be evaluated for retarded time $u = t-r$ and a prime indicates differentiation with respect to the argument u; finally

$$a_{\alpha\beta} \overset{\text{def}}{=} m^{-1}a^{-2}A_{\alpha\beta} \tag{5.7}$$

where $A_{\alpha\beta}$ is the angular momentum (in pseudogalilean coordinates x_i) of the system about O. All the (1s) solutions are of order r^{-1}

$$\overset{(1s)}{B}, \ldots, \overset{(1s)}{J} = \mathrm{O}(r^{-1}) \qquad s \geqslant 0. \tag{5.8}$$

† That $\overset{(1s)}{F} = 0$ for any $s \geqslant 0$ is evident from equation (A.11), since the arbitrary function $\eta(\theta, \phi, u)$ should be chosen as zero, in order that the galilean conditions at spatial infinity be satisfied by the (1s) metric.

It can be verified via lengthy but straightforward calculations that the above (10) and (12) solutions satisfy the (1s) field equations (A.1) to (A.10) with $P = Q = \ldots = W = 0$.

For carrying out the calculations of the next section it is sufficient to retain the terms of order r^{-1} in the (12) solution (5.4). Consequently, on application of this solution to the spinning rod (and ring), only computation of $k_{\alpha\beta}$ from the geometry of the rotating system is required, not of $a_{\alpha\beta}$. From equation (5.6) and the fact that T_{44} therein represents the material density (in the linear approximation), the nonzero $k_{\alpha\beta}$ are readily found to have the following values for the spinning system throughout the interval $0 \leqslant u \leqslant T$ of the spin of the rod:

$$\begin{aligned} k_{11} &= m^{-1}a^{-2}(I\cos^2\omega u + \tfrac{1}{2}I_0) \\ k_{12} &= m^{-1}a^{-2}I\cos\omega u\sin\omega u \\ k_{22} &= m^{-1}a^{-2}(I\sin^2\omega u + \tfrac{1}{2}I_0). \end{aligned} \tag{5.9}$$

In these, I and I_0 denote the moments of inertia, about O, of the rod and ring, respectively.

We now substitute equations (5.5) and (5.9) into equations (5.4) to obtain the appropriate (12) solution for the rod and ring during the period $0 \leqslant u \leqslant T$ of the rotation of the rod. During the substitution we come across terms of order r^{-1} involving expressions of the forms

$$\begin{aligned} &(\cos^2\phi - \sin^2\phi)(\cos^2\omega u - \sin^2\omega u) + 4\cos\phi\sin\phi\cos\omega u\sin\omega u \\ &(\cos^2\phi - \sin^2\phi)\cos\omega u\sin\omega u - \cos\phi\sin\phi(\cos^2\omega u - \sin^2\omega u) \end{aligned} \tag{5.10}$$

as factors. These are easily seen to reduce to

$$\cos(2\omega u - 2\phi) \qquad \tfrac{1}{2}\sin(2\omega u - 2\phi) \tag{5.11}$$

respectively. With these simpler forms adopted, the resulting (12) solution, expressed explicitly to order r^{-1}, turns out to be (for $0 \leqslant u \leqslant T$)

$$\begin{aligned} \overset{(12)}{B} &= -2h\omega^2 r^{-1}(2 - \sin^2\theta)\cos(2\omega u - 2\phi) + O(r^{-3}) \\ \overset{(12)}{D} &= 12h\omega^2 r^{-1}\sin^2\theta\cos(2\omega u - 2\phi) + O(r^{-2}) \\ \overset{(12)}{G} &= -4h\omega^2 r^{-1}\sin\theta\cos\theta\cos(2\omega u - 2\phi) + O(r^{-2}) \\ \overset{(12)}{I} &= 4h\omega^2 r^{-1}\cos\theta\sin(2\omega u - 2\phi) + O(r^{-3}) \\ \overset{(12)}{J} &= -4h\omega^2 r^{-1}\sin\theta\sin(2\omega u - 2\phi) + O(r^{-2}) \end{aligned} \tag{5.12}$$

where

$$h \overset{\text{def}}{=} m^{-1}a^{-2}I. \tag{5.13}$$

Equations (5.2), (5.3), (5.8) and (5.12) are employed in the next section.

6. The (24) approximation

The lowest (2s) approximation in which a secular loss in mass of any isolated coherent source might appear is the (24) one, for the following reason (see also Rotenberg 1964, § 4.7, Bonnor and Rotenberg 1966, Rotenberg 1968b).

The solution of any (ps) approximation is indeterminate to the extent of a *complementary* solution, that is, a solution of

$$\Phi_{lm}(\overset{(ps)}{g_{ik}}) = 0 \tag{6.1}$$

Φ_{lm} standing for the left hand sides of equations (A.1) to (A.10). However, we shall assume that the functions representing the essential sources of the wave field have already been chosen for the 2^s pole waves of the linear ($1s$) approximations

$$\Phi_{lm}(\overset{(1s)}{g_{ik}}) = 0. \tag{6.2}$$

No fresh source functions are to be employed other than those which are required for the satisfaction of the inhomogeneous equations (A.1) to (A.10), which are nonsingular for $r > 0$ and which yield galilean conditions for the metric at spatial infinity.

Reverting to the ($2s$) approximations for $s = 0$ to 4, we note first that the (20) approximation vanishes, since the $\overset{(p0)}{g_{ik}}$ of any ($p0$) approximation is obviously the pth approximation (m^p contribution) to the Schwarzschild–Sachs metric (4.3), and this m^p contribution is absent from this metric when $p \geqslant 2$. The (21) approximation vanishes too, because from equation (5.3), there are no (11) terms to produce with the (10) terms a nonlinear $\overset{(21)}{\Psi_{lm}}$ in equation (3.4). Thus, in accordance with our agreement to withhold new source functions unless they become necessary, $\overset{(20)}{g_{ik}}$ and $\overset{(21)}{g_{ik}}$ must be put zero.

The (22) and (23) approximations are not zero but, because of equations (5.2) and (5.8), they are of too high order of r^{-1} to yield any significant changes of order r^{-1} in the metric (see table 1 of the Appendix).

The (24) approximation is made up of equations (A.1) to (A.10) with the nonlinear quantities $P, \dots, W$ on the right presented in table 2 of the Appendix for the interval $0 \leqslant u \leqslant T$ of rotation of the rod. Use has been made of equations (5.2), (5.3), (5.8) and (5.12) in the compilation of table 2.

By using this table 2 in the solution (A.11) to (A.16) of the approximate field equations (A.1) to (A.10), and by following carefully the procedure outlined in the Appendix with regard to utilizing the solution, we eventually obtain the (24) solution given below for the rod during the interval $0 \leqslant u \leqslant T$ of its spin, and satisfying the (24) approximation to the order indicated in the right column of table 2†.

$$\overset{(24)}{B} = r^{-1}\{h^2\omega^6 u(\tfrac{24}{5}s^2 - \tfrac{4}{15}s^4) - \tfrac{1}{42}h^2\omega^5(s^2+s^4)\sin(4\omega u - 4\phi)\}$$
$$+ h^2\omega^4 r^{-2}\{(8 - 8s^2 + s^4) + s^4\cos(4\omega u - 4\phi)\} + O(r^{-3}) \tag{6.3}$$

$$\overset{(24)}{C} = r^{-1}\{h^2\omega^6 u(-\tfrac{24}{5}s^2 + \tfrac{4}{15}s^4) + \tfrac{1}{42}h^2\omega^5(s^2+s^4)\sin(4\omega u - 4\phi)\}$$
$$+ h^2\omega^4 r^{-2}\{(8 - 8s^2 + s^4) + s^4\cos(4\omega u - 4\phi)\} + O(r^{-3}) \tag{6.4}$$

$$\overset{(24)}{D} = r^{-1}\{\tfrac{64}{5}h^2\omega^6 u - \tfrac{1}{2}h^2\omega^5 s^4 \sin(4\omega u - 4\phi)\}$$
$$+ h^2\omega^4 r^{-2}\{(-4 + 8s^2 - \tfrac{5}{2}s^4) - \tfrac{55}{12}s^4\cos(4\omega u - 4\phi)\} + O(r^{-3}) \tag{6.5}$$

$$\overset{(24)}{F} = -\tfrac{1}{4}h^2\omega^4 r^{-2}\{(8 - 8s^2 + s^4) + s^4\cos(4\omega u - 4\phi)\} + O(r^{-4}) \tag{6.6}$$

† The details have been omitted to save space. These are to be included in a future paper, in which the complete (24) solution will be calculated.

$$\overset{(24)}{G} = h^2\omega^6 r^{-1} u(-\tfrac{48}{5}sc + \tfrac{4}{5}s^3 c) - \tfrac{13}{12}h^2\omega^4 r^{-2} s^3 c \cos(4\omega u - 4\phi) + \mathrm{O}(r^{-3}) \quad (6.7)$$

$$\overset{(24)}{I} = h^2\omega^5 r^{-1}\{\tfrac{12}{5}s^2 c - (\tfrac{1}{42}s^2 c + \tfrac{1}{28}s^4 c)\cos(4\omega u - 4\phi)\} + \mathrm{O}(r^{-3}) \quad (6.8)$$

$$\begin{aligned}\overset{(24)}{J} = {} & h^2\omega^5 r^{-1}\{(\tfrac{24}{5}s - 6s^3) + \tfrac{1}{8}s^5 \cos(4\omega u - 4\phi)\} \\ & + r^{-2}\{-\tfrac{64}{5}h^2\omega^5 us + h^2\omega^4(-\tfrac{13}{12}s^3 + \tfrac{5}{16}s^5)\sin(4\omega u - 4\phi)\} \\ & + \mathrm{O}(r^{-3}) \qquad (6.9)\end{aligned}$$

where $s = \sin\theta$, $c = \cos\theta$.

Disregarding in the above solution static and periodic terms of order r^{-1} and terms of order r^{-n}, $n > 1$, which do not represent any permanent change of order r^{-1} in the metric, we have

$$\begin{aligned} & \overset{(24)}{B} = -\overset{(24)}{C} = h^2\omega^6 r^{-1} u(\tfrac{24}{5}s^2 - \tfrac{4}{15}s^4) \\ & \overset{(24)}{D} = \tfrac{64}{5}h^2\omega^6 r^{-1} u \\ & \overset{(24)}{G} = h^2\omega^6 r^{-1} u(-\tfrac{48}{5}sc + \tfrac{4}{5}s^3 c) \\ & \overset{(24)}{F} = \overset{(24)}{I} = \overset{(24)}{J} = 0. \end{aligned} \qquad (6.10)$$

For $u \leqslant 0$ (before commencement of the spin of the rod) this solution vanishes; thus, as expected, there is nothing of order r^{-1} to add to the initial stationary field. For $u \geqslant T$ (after termination of the spin of the rod) the solution (6.10), when subjected to the coordinate transformation

$$\begin{aligned} r &= r^* + m^2 a^4 h^2 \omega^6 T(-\tfrac{208}{45} + \tfrac{64}{9}s^{*2} - \tfrac{2}{9}s^{*4}) \\ \theta &= \theta^* + m^2 a^4 h^2 \omega^6 T r^{*-1}(\tfrac{208}{45}s^* c^* - \tfrac{4}{45}s^{*3} c^*) \\ \phi &= \phi^* \qquad u = u^* + m^2 a^4 h^2 \omega^6 T(-\tfrac{104}{45}s^{*2} + \tfrac{1}{45}s^{*4}) \end{aligned} \qquad (6.11)$$

with $s^* = \sin\theta^*$, $c^* = \cos\theta^*$, yields the sole term

$$\overset{(24)}{D^*} = \tfrac{64}{5}h^2\omega^6 T r^{*-1} \qquad (6.12)$$

while the lower (ps) solutions are unaltered and the conditions of the Sachs metric remain satisfied. Combining the (24) solution (6.12) (asterisks omitted) with the monopole solution (5.2), we obtain for the final stationary field

$$\mathrm{d}s^2 = -r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \{1 - 2(m - \tfrac{32}{5}I^2\omega^6 T)r^{-1}\}\,\mathrm{d}u^2 + 2\,\mathrm{d}r\,\mathrm{d}u. \qquad (6.13)$$

This metric is of the Schwarzschild–Sachs form (4.3), and in it the expression

$$\Delta m = -\tfrac{32}{5}I^2\omega^6 T \qquad (6.14)$$

clearly shows up as a correction to the Schwarzschild mass of the system, the rod plus the ring.

Suppose μ is the contribution to the change (6.14) of mass of the system due to the smooth transitions for the rod from rest to motion and vice versa. This, being independent of the duration T of the spin of the rod, will clearly account for only an infinitesimal part of Δm in equation (6.14), owing to the shortness of the intervals t_0 and

t_1 of the smooth transitions. Indeed, the rate at which the system loses mass during the constant spin of the rod is, by virtue of equation (6.14)

$$-\frac{\mathrm{d}m}{\mathrm{d}u} = \frac{-\Delta m + \mu}{T} = \frac{32}{5} I^2 \omega^6 + \frac{\mu}{T}. \tag{6.15}$$

So, as $T \to \infty$, equation (6.15) will eventually give

$$-\frac{\mathrm{d}m}{\mathrm{d}u} = \frac{32}{5} I^2 \omega^6. \tag{6.16}$$

We therefore conclude that a rod rotating indefinitely, without any attached device to start and stop its motion, steadily loses mass at the rate (6.16). This accounts for the rate of radiation energy emission given in Eddington (1924 p 251) by the familiar expression on the right of equation (6.16).

Appendix. The approximate field equations for the Sachs metric and their solution

The (ps) approximation for the Sachs metric (4.1), which is the contribution of $m^p a^s$ in equation (1.1) formed by insertion of the expansions (4.4) in equation (1.1), consists of the ten equations below. To save printing the symbols (ps), which ought to have been placed above the capital letters, have been omitted throughout the Appendix.

$$2R_{11} = 0: \ -4r^{-1}F_1 = P \tag{A.1}$$

$$\begin{aligned} 2r^{-2}R_{22} = 0: \ & B_{11} - 2B_{14} + 2r^{-1}(B_1 - B_4 + D_1 - F_1 - G_{12}) \\ & + r^{-2}(-B_{22} + B_{33}\operatorname{cosec}^2\theta - 3B_2\cot\theta + 2B + 2D + 2F_{22} - 4F - 4G_2 \\ & - 2G\cot\theta + 2I_{23}\operatorname{cosec}\theta + 2I_3\operatorname{cosec}\theta\cot\theta - 2J_3\operatorname{cosec}\theta) = Q \end{aligned} \tag{A.2}$$

$$\begin{aligned} 2r^{-2}\operatorname{cosec}^2\theta\, R_{33} = 0: \ & -B_{11} + 2B_{14} + 2r^{-1}(-B_1 + B_4 + D_1 - F_1 \\ & - G_1\cot\theta - J_{13}\operatorname{cosec}\theta) + r^{-2}(-B_{22} + B_{33}\operatorname{cosec}^2\theta - 3B_2\cot\theta + 2B \\ & + 2D + 2F_{33}\operatorname{cosec}^2\theta + 2F_2\cot\theta - 4F - 2G_2 - 4G\cot\theta + 2I_{23}\operatorname{cosec}\theta \\ & + 2I_3\operatorname{cosec}\theta\cot\theta - 4J_3\operatorname{cosec}\theta) = R \end{aligned} \tag{A.3}$$

$$\begin{aligned} 2R_{44} = 0: \ & -D_{11} + 2F_{14} + 2r^{-1}(-D_1 - D_4 + 2F_4 + G_{24} + G_4\cot\theta \\ & + J_{34}\operatorname{cosec}\theta) - r^{-2}(D_{22} + D_{33}\operatorname{cosec}^2\theta + D_2\cot\theta) = S \end{aligned} \tag{A.4}$$

$$\begin{aligned} 2r^{-1}R_{12} = 0: \ & -G_{11} + r^{-1}(-B_{12} - 2B_1\cot\theta + F_{12} - 2G_1 + I_{13}\operatorname{cosec}\theta) \\ & + 2r^{-2}(-F_2 + G) = L \end{aligned} \tag{A.5}$$

$$\begin{aligned} 2R_{14} = 0: \ & -D_{11} + 2F_{14} + r^{-1}(-2D_1 + G_{12} + G_1\cot\theta + J_{13}\operatorname{cosec}\theta) \\ & + r^{-2}(-F_{22} - F_{33}\operatorname{cosec}^2\theta - F_2\cot\theta \\ & + G_2 + G\cot\theta + J_3\operatorname{cosec}\theta) = M \end{aligned} \tag{A.6}$$

$$\begin{aligned} 2r^{-1}R_{24} = 0: \ & -G_{11} + G_{14} + r^{-1}(-B_{24} - 2B_4\cot\theta - D_{12} + F_{12} + F_{24} \\ & - 2G_1 - G_4 + I_{34}\operatorname{cosec}\theta) + r^{-2}(-G_{33}\operatorname{cosec}^2\theta + J_{23}\operatorname{cosec}\theta \\ & + J_3\operatorname{cosec}\theta\cot\theta) = N \end{aligned} \tag{A.7}$$

$$2r^{-1}\operatorname{cosec}\theta R_{13} = 0:\ -J_{11}+r^{-1}(B_{13}\operatorname{cosec}\theta+F_{13}\operatorname{cosec}\theta+I_{12}$$
$$+2I_1\cot\theta-2J_1)+2r^{-2}(-F_3\operatorname{cosec}\theta+J) = U \tag{A.8}$$
$$2r^{-2}\operatorname{cosec}\theta R_{23} = 0:\ -I_{11}+2I_{14}+r^{-1}(-G_{13}\operatorname{cosec}\theta-2I_1+2I_4-J_{12}$$
$$+J_1\cot\theta)+r^{-2}(2F_{23}\operatorname{cosec}\theta-2F_3\operatorname{cosec}\theta\cot\theta-G_3\operatorname{cosec}\theta$$
$$-J_2+J\cot\theta) = V \tag{A.9}$$
$$2r^{-1}\operatorname{cosec}\theta R_{34} = 0:\ -J_{11}+J_{14}+r^{-1}(B_{34}\operatorname{cosec}\theta-D_{13}\operatorname{cosec}\theta$$
$$+F_{13}\operatorname{cosec}\theta+F_{34}\operatorname{cosec}\theta+I_{24}+2I_4\cot\theta-2J_1-J_4)$$
$$+r^{-2}(G_{23}\operatorname{cosec}\theta-G_3\operatorname{cosec}\theta\cot\theta-J_{22}-J_2\cot\theta$$
$$+J\operatorname{cosec}^2\theta) = W. \tag{A.10}$$

Here, a subscript 1, 2, 3 or 4 after B, D, F, G, I or J indicates differentiation with respect to r, θ, ϕ or u, respectively—a notation to apply to any nontensorial symbol, unless otherwise stated or implied. Use has been made of the second of equations (4.1), so that C does not appear in the above equations. The linear terms of these equations all appear explicitly on the left, while the nonlinear terms form the quantities $P, \ldots, W$ on the right. The latter are zero in the linear (1*s*) approximations; in the nonlinear (*ps*) approximations ($p \geqslant 2$), they are determined from solutions of earlier approximations.

It can be shown that from the first seven equations of the above (*ps*) approximation can be derived the following six equations†:

$$F = -\tfrac{1}{4}\int rP\,\mathrm{d}r+\eta(\theta,\phi,u) \tag{A.11}$$
$$\Box' D \overset{\text{def}}{=} D_{11}-2D_{14}+2r^{-1}(D_1+D_4)+r^{-2}(D_{22}+D_2\cot\theta+D_{33}\operatorname{cosec}^2\theta)$$
$$= -S+2(F_1+2r^{-1}F+r^{-2}X)_4 \tag{A.12}$$
$$\Box'' G \overset{\text{def}}{=} r(G_{111}-2G_{114})+(3G_{11}-2G_{14})$$
$$+r^{-1}\{G_{122}+3G_{12}\cot\theta+G_1(\cot^2\theta-1)+G_{133}\operatorname{cosec}^2\theta+2G_4\}$$
$$+r^{-2}\{-G_{22}-3G_2\cot\theta+G(1-\cot^2\theta)-G_{33}\operatorname{cosec}^2\theta\}$$
$$= rL_4-(rN)_1+2D_{11}\cot\theta+F_{112}+2r^{-1}F_{24}$$
$$+\{r^{-2}(X_2+2X\cot\theta)\}_1 \tag{A.13}$$
$$J = \sin\theta\int(rD_1-G_2+r^{-1}X)\,\mathrm{d}\phi-\cos\theta\int G\mathrm{d}\phi+\tau(r,\theta,u) \tag{A.14}$$
$$\Box''' B \overset{\text{def}}{=} B_{11}-2B_{14}+2r^{-1}(B_1-B_4)$$
$$= \tfrac{1}{2}(Q-R)-M-D_{11}+2F_{14}+2r^{-1}(-D_1+G_{12})+2r^{-2}(-F_{22}+G_2) \tag{A.15}$$
$$I = \sin\theta\int\left(\int\{rL+2r^{-1}(F_2-G)\}\,\mathrm{d}r+(rG)_1+B_2-F_2\right)\mathrm{d}\phi$$
$$+2\cos\theta\int B\,\mathrm{d}\phi+\mu(\theta,\phi,u)+\nu(r,\theta,u) \tag{A.16}$$

† The proof will be given in a future paper.

with

$$X \stackrel{\text{def}}{=} \int \{r^2(M-2F_{14})+(F_{22}+F_2\cot\theta+F_{33}\operatorname{cosec}^2\theta)\}\,\mathrm{d}r+\chi(\theta,\phi,u) \tag{A.17}$$

and

$$\eta(\theta,\phi,u),\quad \chi(\theta,\phi,u),\quad \tau(r,\theta,u),\quad \mu(\theta,\phi,u),\quad \nu(r,\theta,u) \tag{A.18}$$

as five functions of integration. This set of equations may be regarded as the formal solution of the (*ps*) approximation; since, F is immediately known from equation (A.11), so that the right hand side of equation (A.12) is readily evaluated. The differential equation (A.12) can be solved for D, to the required degree of accuracy, by using as a trial solution an expansion in ascending powers of r^{-1} for the leading terms. Subsequently, the differential equation (A.13) can be solved for G by a similar procedure, as the terms on the right of this equation are now known. The results so far enable J to be calculated at once from equation (A.14), and the differential equation (A.15) to be solved for B by employment of a trial solution similar to that for D or G. This in turn allows I to be computed immediately from equation (A.16).

During the foregoing process of solution, values for the five arbitrary functions (A.18) and for those emanating from the differential equations (A.12), (A.13) and (A.15) must be chosen with extreme care so that the (*ps*) solution using them in fact satisfies all the ten (*ps*) field equations, including the last three, and is regular for all θ, ϕ and u and for all $r > 0$. It is desirable that the (*ps*) solution also meet the galilean conditions at spatial infinity. However, no additional values are to be assigned to these arbitrary functions, in accordance with the agreement set up in § 6 not to introduce fresh source functions in the nonlinear approximations without a purpose.

Table 1.

Nonlinear terms in the (22) and (23) approximations	Order of error
$P = L = M = U = 0$	0
$Q = R = S = N = V = W = 0$	r^{-4} (at least)

Table 2.

Nonlinear terms in the (24) approximation	Order of error
$P = -2h^2\omega^4r^{-4}\{(8c^2+s^4)+s^4\tilde{c}\}$	r^{-6}
$Q = -R = -8h^2\omega^5r^{-3}s^4\tilde{s}$	r^{-4}
$S = 8h^2\omega^6r^{-2}\{-(8c^2+s^4)+s^4\tilde{c}\}-80h^2\omega^5r^{-3}s^4\tilde{s}$	r^{-4}
$L = 16h^2\omega^4r^{-4}\{(2\cot\theta-5sc+s^3c)+s^3c\tilde{c}\}$	r^{-5}
$M = -4h^2\omega^5r^{-3}s^4\tilde{s}$	r^{-5}
$N = 16h^2\omega^5r^{-3}s^3c\tilde{s}$	r^{-4}
$U = -4h^2\omega^4r^{-4}s^3\tilde{s}$	r^{-5}
$V = 0$	r^{-4}
$W = 24h^2\omega^5r^{-3}\{(4s-3s^3)+s^3\tilde{c}\}$	r^{-4}

Tables 1 and 2 give values for the nonlinear expressions $P, \ldots, W$ in the (22), (23) and (24) approximations, where the notations

$$s = \sin\theta, \quad c = \cos\theta \qquad \tilde{s} = \sin(4\omega u - 4\phi), \quad \tilde{c} = \cos(4\omega u - 4\phi) \qquad \text{(A.19)}$$

have been employed. Tables 1 and 2 refer to the interval $0 \leqslant u \leqslant T$ of rotation of the rod.

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